Level fluctuations in quantum systems with multifractal eigenstates

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The relationship between universality of level fluctuation laws and multifractality of eigenstates is studied by analyzing energy spectra of tight-binding models of quantum billiards with multifractal eigenstates. We show that level fluctuations in our models are well described by the universal statistical laws such as Poisson statistics for integrable systems and statistics for the Gaussian orthogonal random matrix ensemble as long as the energy levels are located in bandlike spectra, which indicates that level statistics is irrelative to the multifractal properties of eigenstates. Our results are further confirmed by statistical properties of energy spectra in tight-binding models of two-dimensional quasicrystals. [S1063-651X(99)07204-9]

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The random matrix theory (RMT), initially proposed by Wigner and Dyson [1] for studying the spectrum of complex nuclei, is now widely applied to many physical problems such as disordered metals, mesoscopic systems and chaotic quantum billiards [2–6]. It has been shown that level fluctuations of these complex systems are described by universal laws such as statistics for the Gaussian orthogonal random matrix ensemble (GOE) [2]. On the other hand, it has been conjectured that level fluctuations for integrable quantum billiards follow the Poisson statistics in most cases [7,8].

Eigenstates of the above quantum billiards and of GOE are extended, i.e., the averaged spatial extension of the eigenstate scales with the size of the system. Multifractal eigenstates occur in various systems, e.g., at the metal-insulator transition (MIT) in the three-dimensional (3D) Anderson model of disorder [9] and in tight-binding (TB) models of quasicrystals [10,11]. The multifractality of an eigenstate $\psi_E(\mathbf{r})$ is revealed by the scaling behavior of the inverse participation ratio [12] $\mathcal{P}(E,V)=\Sigma_{\mathbf{r}}|\psi_E(\mathbf{r})|^4 \sim V^{-\alpha(E)}$, where V is the size of the system. A multifractal eigenstate is characteristic of $0<\alpha(E)<1$, which scales differently from the extended state with $\alpha(E)=1$. A natural question is: what is the underlying level statistics in quantum systems with multifractal eigenstates? Extensive studies [13–16] for the level fluctuations at the MIT of the Anderson model indicate that there exists a new kind of level statistics different from the known level fluctuation laws, although controversial results have been obtained concerning the specific form of the new statistics. It has been found that level fluctuations for Coulomb billiards [17], rough circular billiards [18], and pseudointegrable rational billiards [19] also show different statistics from the GOE result and the Poission statistics. Much attention has been paid to understanding the mechnism of the new statistics [15–17,20]. It is conjectured that [15–17,20] level distributions are strongly influenced by the spatial overlapping of the corresponding eigenstates, and thus the new statistics can be described in terms of the multifractality of eigenstates. However, a recent calculation [21] showed that the statistical properties of energy spectra of a TB model of 2D quasicrystals are well described by the GOE. These two different results indicate that the relationship between level fluctuations and multifratcality of eigenstates is not yet clear.

In this paper, we investigate level fluctuations of spectra in TB models of quantum billiards with multifractal eigenstates. We note that multifarctal eigenstates may be associated with a bandlike or a multi-fractal energy spectrum [10,11,24,25]. Consider an energy interval δ at energy E. In general, the number of levels ΔN inside the interval has scaling behavior $\Delta N \sim \delta^{\beta(E)}$, as $\delta \to 0$. Bandlike spectra have $\beta(E)\equiv 1$. $0<\beta(E)<1$ corresponds to a multifractal spectrum. It is evident that the spectrum of a system of finite size is composed of discrete energy levels. The bandlike and the multifractal behaviors of the spectra are defined when the system size goes to infinity. We will show that there is a class of quantum systems whose level fluctuations are irrelative to the multifractality of the eigenstates as long as the energy levels are located in bandlike spectra.

Our study is based on the 2D Fibonacci lattice [24,25] defined on a square lattice of unit cell with TB Hamiltonian $H=\Sigma_i \epsilon_i |i\rangle\langle i| + \Sigma_{ij} t_{i,j} |i\rangle\langle j|$, where ϵ_i is the site potential at ith site and $t_{i,j}\equiv 1$ are the nearest-neighbor hopping integrals. Potentials ϵ_i are assumed to be separable $\epsilon_i=\epsilon_{ix}+\epsilon_{iy}$, where $\epsilon_{ik}(k=x,y)$ independently forms the Fibonacci sequence with two kinds of values $u>0$ and $-u$. The Fibonacci sequence $S_l(l\to\infty)$ is given by the recusion relation $S_{l+1}=S_l+S_{l-1}$ $(l\geq 1)$ with initial condition $S_0=B$ and $S_1=A$, here A and B correspond to potentials $-u$ and u, respectively. Successively, one has $S_0=B$, $S_1=A$, $S_2=AB$, $S_3=ABA$, $S_4=ABAAB,\ldots,$ and $S_\infty=ABAABABAABAAB\ldots$. This model has been widely used to study electronic properties of quasiperiodic systems [11,24,25]. Let us consider 2D Fibonacci lattices with square boundary shape $0\leq x\leq L$, $0\leq y\leq L$ and with Sinai's billiard boundary shape which is given by definition $0\leq y\leq x\leq L$ and $x^2+y^2\geq L^2/4$ [2,4]. We mention that when $u\equiv 0$, poten-

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tials inside the lattices are homogeneous and eigenstates are extended. Therefore one has the TB version of the integrable square billiard and the chaotic Sinai's billiard. In this case, level fluctuations are described by the Poisson statistics and the GOE statistics, respectively.

For the above generalized square billiard with Fibonacci potentials, one can show that [24,25] the energy level E and its corresponding eigenstate $\psi_E(x,y)$ are given by $E=E_x+E_y$ and $\psi_E(x,y)=\phi_{E_x}(x)\times\phi_{E_y}(y)$, where E_k and $\phi_{E_k}(k)$ $(k=x,y)$ are the eigenenergy and the eigenstate of 1D Fibonacci chain, respectively. It follows from this relation that [24,25] the spectrum is bandlike for $u<0.6$, multifractal for $u>2$, or a mixture for $0.6<u<2$ with some parts bandlike and some parts multifractal. We can show that eigenstates are multifractal $\mathcal{P}(E,N)\sim N^{-\alpha(E)}$ with $2\alpha(E)=\alpha_{1D}(E_x)+\alpha_{1D}(E_y)$, where N is the number of sites of the billiard and $\alpha_{1D}(E_k)$ is the scaling exponent in 1D Fibonacci chain with $0<\alpha_{1D}(E_k)<1$ for any $u>0$ due to the multifractality of the eigenstates [10,11]. Our numerical calculation shows that the spectra and the eigenstates of the generalized Sinai's billiard with Fibonacci potential have the same properties as that of the generalized square billiard.

In general, the density of states (DOS) varies with energy E. In order to observe the universal level fluctuation, one needs to unfold the spectrum to have a linear integrated DOS (IDOS) [2]. To fulfill this one can replace levels E_i by $e_i=N_{\mathrm{av}}(E_i)$, where N_{av} is the smoothed IDOS which can be obtained by fitting the IDOS to a cubic spline or by local density average. We emphasize that IDOS of a multifractal spectrum exhibits nonuniform local nonlinear behaviors $\Delta N(E)\propto\delta^{\beta(E)}$ $(\delta\to 0)$ with $\beta(E)<1$. Thus the traditional unfolding procedure on the basis of a local linear behavior of the IDOS is not applicable for multifractal spectra. Therefore we focus our attention on the energy levels located in bandlike spectra in order to compare our results with the traditional universal level fluctuation laws.

The most frequently used quantities describing level fluctuations are the level spacing distribution (LSD) $P(s)$, number variance Σ_2, and spectral rigidity Δ_3. $P(s)$ is the probability density of level spacings $s=e_{i+1}-e_i$. One may also consider the integrated LSD (ILSD) $I(s)=\int_s^\infty P(s')ds'$ which is numerically more stable. Σ_2 measures the fluctuation of the number of levels ΔN in a strip of width W, $\Sigma_2(W)=\langle\Delta N^2\rangle-\langle\Delta N\rangle^2$, where $\langle\ \rangle$ denotes the spectral average. Δ_3 is defined as the least squares deviation of $N(\epsilon)$ from a linear behavior, averaged over a range K,

$$\Delta_3(K)=\left\langle \min_{A,B}(1/K)\int d\epsilon'[N(\epsilon')-A\epsilon'-B]^2\right\rangle,$$

with integral interval $\epsilon-K/2\leqslant\epsilon'\leqslant\epsilon+K/2$. For Poisson statistics, we have $P(s)=\exp(-s)$, $\Sigma_2(W)=W$, and $\Delta_3(K)=K/15$. For GOE, there is no expression for $P(s)$ in a closed form while RMT gives $\Sigma_2(W)\sim(2/\pi^2)\ln(W)$ and $\Delta_3(K)\sim(1/\pi^2)\ln(K)$ for $W\gg 1$ and $K\gg 1$.

In general, scaling exponent $\alpha(E)$ varies with energy [9–11,22,23]. Numerically, one [23] analyzes the scaling behavior of $\mathcal{P}(E,N)$ at energy E by averaging the $\mathcal{P}(E,N)$ data

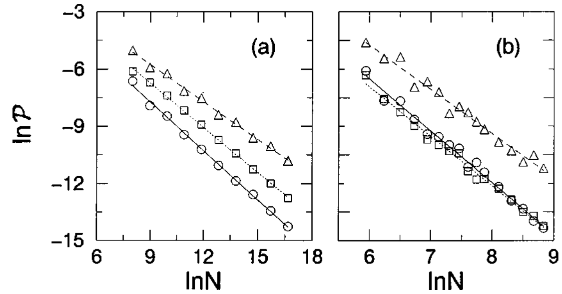

FIG. 1. Scaling behavior $\mathcal{P}(E,N)\propto N^{-\alpha(E)}$ for (a) generalized square billiards and (b) generalized Sinai billiards with Fibonacci potentials u. Circles (○), boxes (□), and triangles (△) are results at energies $E=E_{\mathrm{L}}$ for $u=0.5$, $E=0$ for $u=0.5$, and $E=-0.91$ for $u=1$, respectively. Lines indicate the results of the least-squares fit.

over a small energy interval $|E'-E|\leqslant\Delta E$. Apparently, a reliable result should be independent of the value of ΔE. According to this, we numerically found that the scaling behavior $\mathcal{P}(E,N)\sim N^{-\alpha(E)}$ with $\alpha(E)<1$ holds for the models we studied. As examples, $\mathcal{P}(E,N)$ at the lower band edge E_L and at energies $E=0$ and $E=-0.91$ close to the band center is illustrated in Fig. 1, where $\Delta E=0.001$ for the generalized square billiard and $\Delta E=0.05$ for the generalized Sinai's billiard. The later interval is larger due to the smaller system sizes but it is still much smaller than the bandwidth 8.32 and 9.21 for $u=0.5$ and $u=1$, respectively. For the generalized square billiard, $\alpha(E_{\mathrm{L}})\approx 0.85$, $\alpha(0)\approx 0.78$ for $u=0.5$, and $\alpha(-0.91)\approx 0.65$ for $u=1$. For the generalized Sinai's billiard, one finds $\alpha(E_{\mathrm{L}})\approx 0.90$, $\alpha(0)\approx 0.86$ for $u=0.5$, and $\alpha(-0.91)\approx 0.77$ for $u=1$. We mention that we have also calculated $\mathcal{P}(E,N)$ with different small intervals ΔE and found the same conclusion.

Figure 2 shows statistics for energy levels around the energies where eigenstates exhibit multifractal behavior as shown in Fig. 1. Apparently, although eigenstates are multi-

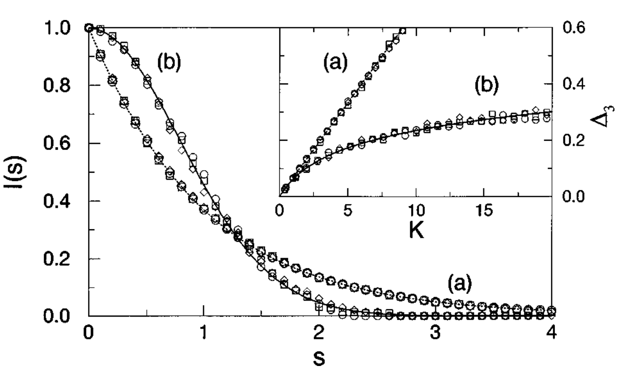

FIG. 2. Statistical properties of energy levels in various parts of energy spectra of (a) the generalized square billiard of size $N=987\times 987$ and (b) the generalized Sinai billiard of size $R=130,N=6870$. Solid line and dotted line indicate the GOE statistics and the Poisson statistics, respectively. Circles (○), boxes (□) and diamonds (◇) in (a) are results for $E_{\mathrm{L}}\leqslant E\leqslant E_{\mathrm{L}}+0.01$ (60 245 levels, $u=0.5$), $-0.01\leqslant E\leqslant 0.01$ (2433 levels, $u=0.5$), and $-0.97\leqslant E\leqslant -0.85$ (9927 levels, $u=1$), respectively. Circles (○), boxes (□) and diamonds (◇) in (b) are results for $E_{\mathrm{L}}\leqslant E\leqslant E_{\mathrm{L}}+0.4$ (235 levels, $u=0.5$), $-0.1\leqslant E\leqslant 0.1$ (289 levels, $u=0.5$), and $-0.97\leqslant E\leqslant -0.85$ (195 levels, $u=1$), respectively.

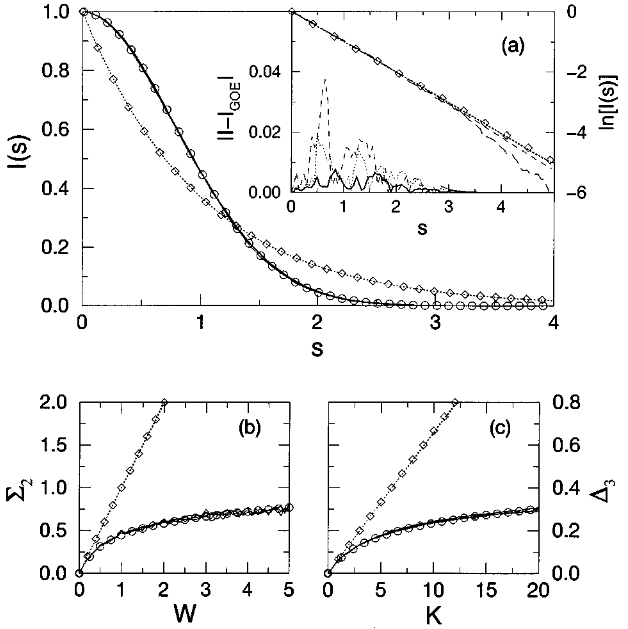

FIG. 3. Statistical properties for energy spectra of the generalized square billiards (dotted lines) of size $N=987\times 987$ and of the generalized Sinai billiards (solid lines) of size $R=130, N=6870$ with Fibonacci potentials $u=0.1$, 0.2, 0.3, 0.4, and 0.5. Diamonds ($\diamond$) and circles ($\bigcirc$) indicate the Poisson statistics and GOE statistics, respectively. The upper curves in the inset of (a) illustrate $\ln[I(s)]$ for the generalized square billiards of sizes $N=987\times 987$ (dotted line), 377×377 (dashed line), and 55×55 (long dashed line). The lower curves in the inset of (a) illustrate deviations $|I(s)-I_{GOE}(s)|$ for the generalized Sinai billiards of sizes $R=130, N=6870$ (solid line), $R=55, N=1247$ (dotted line), and $R=34, N=486$ (dashed line).

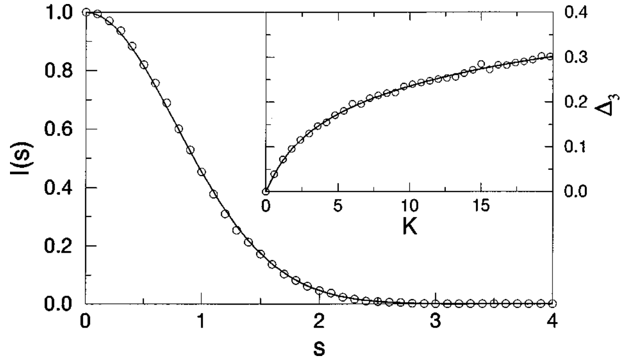

FIG. 4. Statistical properties of energy levels in a range $-0.5\leqslant u\leqslant 0.5$ for the octagonal tight-binding model with $t=0.7$ (circles $\bigcirc$). The system size is $L=80, N=7785$. The solid lines are the results of GOE.

fractal, the level fluctuations are well described by the Poisson statistics and the GOE statistics for the generalized square billiard and the generalized Sinai's billiards, respectively. In order to further support our conclusion, we consider level statistics for the whole spectrum. In Fig. 3, we present results for $u=0.1$, 0.2, 0.3, 0.4, and 0.5. Again, one finds a very good agreement with the Poisson statistics and the GOE statistics. Level fluctuations for systems of different sizes are shown in inset of Fig. 3(a). It is easy to see that the deviations from the Poisson distribution and the GOE distribution turn to be smaller for systems with larger sizes, which indicates that the observed statistics is size independent. Some parts of the spectrum for $0.6<u<2$ are multifractal but some remain bandlike. We find that fluctuations of energy levels in the bandlike region are still well described by the traditional universal statistical laws. Such behavior can be seen in Fig. 2 for energy levels in a range $[-0.97, -0.85]$ for $u=1$, where we find the bandlike behavior. In conclusion, our results for the generalized billiards with multifractal eigenstates clearly show that the level statistics is irrelevant to the multifractality of the eigenstates.

The above conclusion also holds for other models. Let us consider a TB Hamiltonian [26,27] $H=\Sigma_i[(1-t)z_i|i\rangle\langle i|+\Sigma_j^{z_i}t|i\rangle\langle j|]$ defined on the quasiperiodic octagonal tiling, where $0<t\leqslant 1$ denotes the hopping integral, and z_i is the coordination number of ith site which takes six kinds of values 3, 4, 5, 6, 7, and 8. It was found [27] that the spectrum for $t=1$ is bandlike and the spectra for $t_c<t<1$ ($t_c\approx 0.35$) exhibit bandlike behavior in some energy regions. Eigenstates for $0<t\leqslant 1$ are multifractal in general, associated with an anomalous diffusion [27]. It was well illustrated [21] that the underlying level statistics for $t=1$ is described by GOE. However, explicit description of the level fluctuations in the general case $0<t<1$ [27,28] is still lacking. By analyzing energy spectra of the octagonal patches introduced in Ref. [21], we find that the statistical properties of energy levels in the bandlike region for $t_c<t<1$ are well described by GOE as it does for $t=1$. As an example, Fig. 4 presents level statistics for energy levels in a range $-0.5\leqslant E\leqslant 0.5$ for $t=0.7$ on a patch without exact symmetry given by [21] $0\leqslant x\leqslant L$, $-(L/4)\leqslant y\leqslant(3L/4)$ (the eightfold symmetrical site is located at the origin of the xy plane). We note that our calculation shows that $\mathcal{P}(E,N)$ for energies around $E=0$ exhibits a scaling behavior with $\alpha(E)\approx 0.84$. From Fig. 4, we can see that ILSD $I(s)$ and spectral rigidity Δ_3 agree quite well with the GOE results.

Finally, we mention that we have studied level statistics for energy levels in multifractal spectra without unfolding. It has been shown [29] that the level statistics of a multifractal spectrum is characteristic of an inverse power law of the LSD $P(s)\sim s^{-\gamma}$ with $\gamma>0$. Such a behavior was found [29] in the Harper model and also in the multifractal spectra of the octagonal TB models [27,28]. Our calculations for both the generalized billiards with Fibonacci potentials and the octagonal TB models further confirm the inverse power law.

In summary, we study level fluctuations of TB models of quantum billiards with 2D Fibonacci potentials. The scaling behavior of the inverse participation ratio indicates that the corresponding eigenstates are multifractal. We note that the widely used unfolding procedure to find universal level fluctuations is not applicable for multifractal spectra. Taking this into account, we show that level fluctuations for energy spectra of the generalized square billiard and the generalized Sinai's billiard with Fibonacci potentials are well described by the Poisson statistics and the GOE statistics, respectively, as long as the energy levels are located in bandlike spectra in the limit of the infinite system size. We also show that such a conclusion holds for TB models of 2D octagonal quasicrystal. Our results show that, in contrast to the case of the critical level statistics observed at the MIT of the Anderson model [13–16], there is another class of quantum systems with multifractal eigenstates whose level statistics is irrelative to the multifractality of the eigenstates.

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